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DISCONTINUOUS SOLUTIONS OF SEMILINEAR DIFFERENTIAL-ALGEBRAIC EQUATIONS PART I: DISTRIBUTION SOLUTIONS

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DISCONTINUOUS SOLUTIONS OF SEMILINEAR DIFFERENTIAL-ALGEBRAIC EQUATIONS. PART I: DISTRIBUTION SOLUTIONS[1]

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ABSTRACT. There is strong physical evidence that a full treatment of differential-algebraic equations should incorporate solutions with jump discontinuities. It is shown here that for semilinear problems the setting of distributions allows for the development of a theory where indeed such discontinuities may occur. This approach also settles the problem of inconsistent initial conditions in a very simple way. On the other hand, new issues arise as not only uniqueness, but even countability of the number of solutions of initial value problems may now be lost. A physically motivated but purely mathematical selection procedure to overcome this difficulty is discussed in Part II of this paper.

1. Introduction.

The investigation of quasilinear differential-algebraic equations (DAE's) in $\mathbb{R}^n$,

$$A(t,x)\dot{x} = G(t,x), \tag{1.1}$$

given in [RRh2] and [RRh3] (see also [CD], [SDe] and [T]) has revealed the presence of singularities, notably impasse points, beyond which classical solutions cannot be continued. Thus, under the assumption that the system (1.1) governs the evolution of the state variable x at all times, discontinuous and hence nonclassical solutions of (1.1) may play a key role in such problems.

For linear time-dependent problems

$$A(t)\dot{x} + B(t)x = b \tag{1.2}$$

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where $b \in (\mathcal{D}'(J))^n, J \subset \mathbb{R}$ is an open interval, and $\dot{x}$ denotes the derivative of x in the sense of distributions, a rather complete theory can be developed ([RRh4]). More specifically, when b is "almost" a function, initial value problems associated with (1.2) continue to make sense and have a unique solution ([RRh5]). The admissible class of distributions for the validity of such results includes functions with jump discontinuities. It turns out that for index-one problems, the solutions are also functions with jump discontinuities (i.e., not classical ones) while for higher index problems they may also exhibit an "impulsive" part; that is, contain a linear combination of derivatives of Dirac delta distributions. It is noteworthy that the jumps in the solutions are calculable, and hence that these discontinuous solutions are characterized as completely as the classical ones.

It becomes evident that an attempt at resolving the ambiguity created by the presence of impasse points in (1.1) should be made via the concept of distribution solution. This is the topic of this paper for the special case of semilinear DAE's of index one

$$A(t)\dot{x} = G(t, x). \tag{1.3}$$

The restriction to this case is justified by the following consideration: If A is sufficiently smooth, the product $A(t)\dot{x}$ is well defined for any distribution x, whereas an expression such as $A(x)\dot{x}$ usually makes no sense even when x is a piecewise smooth function with a jump discontinuity. Indeed, the discontinuity in x induces a discontinuity in $A(x)$ which therefore cannot be multiplied by the Dirac delta distribution in $\dot{x}$ arising from the jump of x. There are ways to circumvent (in part) this difficulty, but only at the expense of additional technicalities that we prefer not to consider at this time.

Fortunately, the restriction to semilinear problems does not significantly affect the range of applications, as many concrete problems with impasse points or other singularities even involve a constant matrix A in (1.3). But it is important that the problem be of index one, for the semilinear structure is not preserved by the reduction procedures involved in higher index problems.

Remark 1.1: In fact, no reduction procedure described in the literature applies without restriction to distribution solutions, except that given in [RRh4] for the linear case. The

other general techniques (see [RRh1], or [CG] developing another point of view based on augmented systems) implicitly use the fact that the solutions and their derivatives have pointwise values, and hence are not applicable to distributions as they stand. □

Usually, nonautonomous problems such as (1.3) are more conveniently handled within the framework of autonomous systems by changing the variable x into (t, x) and adding the equation $\dot{t} = 1$. But because this transforms a semilinear problem into a quasilinear one, it is no longer appropriate to do so when solutions are sought in the sense of distributions. Since this work relies rather heavily upon prior results established for classical solutions and hence developed mostly for autonomous problems (for the reason just mentioned), it is important to rephrase these results for the case of the nonautonomous system (1.3), thus making it explicit how the variable t is involved in various places, notably in the definition of impasse points. This is done in the next section, where some additional properties are also highlighted, as they become important later (but were not involved, hence not emphasized, in the treatment of classical solutions).

Section 3 deals with the discontinuous solutions of (1.3). As expected, the setting of distributions allows the solutions to jump at impasse (and other) points. But this pleasant conclusion comes along with another, much less welcome one: Once the setting of distributions has been adopted and discontinuities have become possible, they abuse the opportunity. Jumps may also occur in places where there was no reason to expect them before, and simple examples show that, now, initial value problems may have uncountably many solutions. It is the purpose of Part II of this paper to show how this difficulty may be overcome. Meanwhile, in Section 4 it is shown that distribution solutions also allow for a simple answer to the well known question regarding "inconsistent" initial values.

Remark 1.2: Any discussion of the discontinuous solutions of (1.3) must include jumps at singularities since this is where jumps are widely known to occur usually. Then the justification of the distribution approach requires precise information about the behavior of the classical solutions of (1.3) in the vicinity of singularities. For most singularities, this information is still lacking, but in the case of impasse points it is hidden in the (combined) works [R] and [RRh2]. This is why our discussion of jumps at singularities is

limited to impasse points, and presumably one of the reasons why the very natural idea of introducing discontinuous solutions of (1.3) via distributions has not appeared much earlier in the literature. □

The "glut" of distribution solutions appears to be a serious shortcoming of the distribution approach, which has no analog for linear problems. In the second part of this paper, we try to resolve this shortcoming, without, of course, repudiating the concept. Evidently, depending upon the "physical" setting, many of the unwanted distribution solutions may be ruled out for lack of relevance, but such a decision is based upon a non-mathematical argument. In Part II of the paper, we will show that a selection can be made according to physically motivated but purely mathematical criteria. The idea is simply that acceptable solutions must be consistent (in a way to be defined) with a given class of perturbations. In practice, the admissible perturbations are dictated by the physical origin of the problem. This concept of consistency, called "$\mathcal{P}$-consistency", where $\mathcal{P}$ stands for "perturbation", makes standard and novel connections with singular perturbation theory. Standard because the underlying idea is that solutions of the unperturbed problem should be "approximated", locally at least, by solutions of the perturbed problems, and novel because the known criteria for this property to be true are used to sort out the solutions of the unperturbed problem, and to discard many of the spurious distribution solutions.

Naturally, almost all work devoted to the discontinuous solutions of (1.3) stresses connections with singular perturbation theory (with various degrees of emphasis). However, to summarize, the general trend has been to *define* the discontinuous solutions of (1.3) as pointwise or other limits of solutions of ODE perturbations (see e.g. [SDe]). In sharp contrast, the distribution approach permits to consider such discontinuous solutions without any reference to perturbations. Perturbations become involved when it comes to selecting the meaningful solutions, but then the approximation criterion, difficult to check in practice, may be replaced by a weaker and much more convenient eigenvalue condition. Furthermore, DAE rather than ODE perturbations can be used for the selection procedure. This is especially useful, as a number of physically motivated perturbations arise as DAE's and not ODE's.

Expanded examples (from electrical network theory) are given at the end of Part II. Since these examples also illustrate various points made in this first part, we have not included here any other (physically motivated) examples.

2. Geometrically Nonsingular Semilinear DAE's of Index One.

The material presented in this section is in part condensed from the articles [RRh1] and [RRh2] and specialized to the case of semilinear DAE's; that is, to problems of the form

$$A(t)\dot{x} = G(t,x), \tag{2.1}$$

where $A \in C^\infty(J;\mathcal{L}(\mathbb{R}^n))$, $G \in C^\infty(J \times \mathbb{R}^n;\mathbb{R}^n)$ and $J \subset \mathbb{R}$ is an open interval. Consistent with the requirements that (2.1) is a DAE and not an explicit ODE, we shall assume throughout that

$$\text{rank } A(t) = r\ , \quad 0 \leq r < n, \quad \forall t \in J. \tag{2.2}$$

Furthermore, since all the results in this paper involve only local assumptions in the "time" variable t, it is not restrictive to suppose that the interval J has been shrunk so that there exists a common complement Z of dimension $n-r$ to all the spaces rge $A(t), t \in J$:

$$\mathbb{R}^n = \text{ rge } A(t) \oplus Z, \quad \forall t \in J. \tag{2.3}$$

We shall call $Q(t) \in \mathcal{L}(\mathbb{R}^n, Z)$ the projection onto Z associated with the decomposition (2.3). By elementary arguments it follows that

$$Q \in C^\infty(J;\mathcal{L}(\mathbb{R}^n, Z)). \tag{2.4}$$

As is well-known, condition (2.2) alone is not sufficient to provide a satisfactory existence theory for the classical solutions of the DAE (2.1). Accordingly, we assume also that

$$\begin{cases} \text{The mapping} \\ (t,x) \in J \times \mathbb{R}^n \longmapsto Q(t)G(t,x) \in Z \\ \text{is a submersion at each point of its zero set.} \end{cases} \tag{2.5}$$

Condition (2.5) is readily seen to be independent of the space Z in (2.3), and it implies at once that the set

$$W = \{(t,x) \in J \times \mathbb{R}^n : Q(t)G(t,x) = 0\} = \{(t,x) \in J \times \mathbb{R}^n : G(t,x) \in \text{ rge } A(t)\} \tag{2.6}$$

is a closed $(r+1)$-dimensional C^∞ submanifold of $J \times \mathbb{R}^n$ (also independent of Z). Now, the mapping

$$(t,x,p) \in J \times \mathbb{R}^n \times \mathbb{R}^n \longmapsto A(t)p - G(t,x) \in \mathbb{R}^n, \tag{2.7}$$

with the derivative

$$(\tau,h,q) \in \mathbb{R} \times \mathbb{R}^n \times \mathbb{R}^n \longmapsto \tau D_tA(t)p + A(t)q - \tau D_tG(t,x) - D_xG(t,x)h \in \mathbb{R}^n. \tag{2.8}$$

is a submersion at each point (t,x,p) of its zero set. Indeed, we have $(t,x) \in W$ and hence for every $u \in \mathbb{R}^n$ there is (by (2.5)) a $(\tau,h) \in \mathbb{R} \times \mathbb{R}^n$ such that $\tau D_tQ(t)G(t,x) + \tau Q(t)D_tG(t,x) + Q(t)D_xG(t,x)h = -Q(t)u$. From the identity $Q(t)A(t) \equiv 0$, we infer that $D_tQ(t)A(t) = -Q(t)D_tA(t)$. Together with the relation $G(t,x) = A(t)p$, this yields $D_tQ(t)G(t,x) = -Q(t)D_tA(t)p$, and hence

$$u - \tau D_tA(t)p + \tau D_tG(t,x) + D_xG(t,x)h \in \text{ rge } A(t),$$

i.e. there is $q \in \mathbb{R}^n$ such that

$$\tau D_tA(t)p + A(t)q - \tau D_tG(t,x) - D_xG(t,x)h = u.$$

Since $u \in \mathbb{R}^n$ is arbitrary, it follows that (2.8) is surjective. (The converse is also true: If (2.8) is surjective at the points of the zero set of (2.7), then (2.5) holds. The proof is trivial modulo the remark that $D_tQ(t)G(t,x) = -Q(t)D_tA(t)p$ already used before.)

It follows from all this that the set

$$\{(t,x,p) \in J \times \mathbb{R}^n \times \mathbb{R}^n : A(t)p - G(t,x) = 0\}$$

is a closed $(n+1)$-dimensional C^∞-submanifold of $J \times \mathbb{R}^n \times \mathbb{R}^n$, and hence that the set

$$M = \{(t,x,1,p) \in J \times \mathbb{R}^n \times \mathbb{R} \times \mathbb{R}^n : A(t)p - G(t,x) = 0\} \tag{2.9}$$

is a closed $(n+1)$-dimensional C^∞ submanifold of $J \times \mathbb{R}^n \times \mathbb{R} \times \mathbb{R}^n$. Identifying $J \times \mathbb{R}^n \times \mathbb{R} \times \mathbb{R}^n \simeq T(J \times \mathbb{R}^n)$ (tangent bundle of the first two factors) we see that M can be viewed as a submanifold of $T(J \times \mathbb{R}^n)$. On the other hand, from the embedding $W \subset J \times \mathbb{R}^n$ we infer that $TW \subset T(J \times \mathbb{R}^n)$, and the set-theoretic intersection $TW \cap M$ makes sense. Note also that

$$W = \Pi(M), \tag{2.10}$$

where $\Pi : T(J \times \mathbb{R}^n) \to J \times \mathbb{R}^n$ is the canonical projection.

Definition 2.1. *The pair $(t,x) \in J \times \mathbb{R}^n$ is consistent with the DAE (2.1) if $(t,x) \in \Pi(TW \cap M) \subset W$ (see (2.10)). Accordingly,*

$$W^c := \Pi(TW \cap M) \subset W, \tag{2.11}$$

is called the set of consistent points for the DAE (2.1).

The third and final assumption needed to obtain an existence and uniqueness theory for the DAE (2.1) is

$$(t,x) \in W^c \Rightarrow Q(t)D_xG(t,x)_{|\ker A(t)} \in GL(\ker A(t), Z). \tag{2.12}$$

It is once again straightforward to check that condition (2.12) is independent of the choice of the space Z in (2.3). A useful, equivalent formulation of condition (2.12) is contained in the following proposition.

Proposition 2.1. *We have*

$$\tilde{W}^c := \{(t,x) \in W : Q(t)D_xG(t,x)_{|\ker A(t)} \in GL(\ker A(t), Z)\} \subset W^c. \tag{2.13}$$

In particular, condition (2.12) holds if and only if

$$W^c = \tilde{W}^c. \tag{2.14}$$

As a result, if (2.12) holds, then W^c is an open subset of W.

Proof. Let $(t,x) \in \tilde{W}^c$ be given. Since $\tilde{W}^c \subset W$, there exists a $p \in \mathbb{R}^n$ such that $A(t)p = G(t,x)$. Because QG takes values in Z, we have $D_t(QG)(t,x) \in Z$, and by

the surjectivity of $Q(t)D_xG(t,x)_{|\ker A(t)}$ there is a $k \in \ker A(t)$ such that $Q(t)D_xG(t,x)k = -Q(t)D_xG(t,x)p - D_t(QG)(t,x)$; that is,

$$D_t(QG)(t,x) + Q(t)D_xG(t,x)(p+k) = 0. \tag{2.15}$$

Now, from (2.5) and (2.6),

$$T_{(t,x)}W = \ker D(QG)(t,x) = \{(\tau,h) \in \mathbb{R} \times \mathbb{R}^n : \tau D_t(QG)(t,x) + Q(t)D_xG(t,x)h = 0\},$$

so that relation (2.15) also reads $(1, p+k) \in T_{(t,x)}W$ and hence $(t,x,1,p+k) \in TW$. Also, $(t,x,1,p) \in M$ since $A(t)(p+k) - G(t,x) = A(t)p - G(t,x) = 0$ (see (2.9) and recall $k \in \ker\, A(t)$). Thus, $(t,x,1,p+k) \in TW \cap M$ and therefore $(t,x) \in \Pi(TW \cap M) = W^c$.

Since (2.12) amounts to the inclusion $W^c \subset \tilde{W}^c$, the above proves the equivalence of (2.12) and (2.14). An elementary contradiction argument shows that $\tilde{W}^c$ is open in W. □

Definition 2.2. *The semilinear DAE (2.1) is geometrically nonsingular of index* 1 *if the conditions (2.2), (2.5) and (2.12) (or, equivalently, (2.14)) hold.*

The relevance of these concepts to the existence of classical solutions of the DAE (2.1) is provided by Theorem 2.1 below. The given proof only establishes the connection with more general results in [RRh1] or [RRh2] from which it can be derived.

Theorem 2.1. *Let the DAE (2.1) be geometrically nonsingular of index* 1*. Then:*

(i) If $x \in C^1(J;\mathbb{R}^n)$ is a solution of (2.1), we have $(t, x(t)) \in W^c$, for all $t \in J$.

(ii) Conversely, given $(t_0, x_0) \in W^c$ and after shrinking J about t_0 if necessary, there is a unique solution $x \in C^1(J;\mathbb{R}^n)$, actually of class C^∞, such that $x(t_0) = x_0$.

Proof. By adding the equation $\dot{t} = 1$ and setting $\tilde{x} = (t,x)$, the DAE (2.1) is transformed into the quasilinear DAE

$$\tilde{A}(\tilde{x})\dot{\tilde{x}} = \tilde{G}(\tilde{x}), \tag{2.16}$$

where

$$\tilde{A}(\tilde{x}) = \begin{pmatrix} 1 & 0 \\ 0 & A(t) \end{pmatrix} \quad , \quad \tilde{G}(\tilde{x}) = \begin{pmatrix} 1 \\ G(t,x) \end{pmatrix}, \tag{2.17}$$

and the initial condition $x(t_0) = x_0$ becomes

$$\tilde{x}(t_0) = (t_0, x_0). \tag{2.18}$$

Conditions (2.2), (2.5) and (2.12) amount to saying that the autonomous quasilinear DAE (2.16) is geometrically nonsingular of index 1 in the sense of [RRh2]. Here it is worth mentioning that condition (2.12) is equivalent with

$$\tilde{x} \in W^c \Rightarrow \text{ rank } \tilde{A}(\tilde{x})_{|T_{\tilde{x}}W} = r + 1 (= \text{ rank } \tilde{A}(\tilde{x})),$$

which follows from the characterization $T_{\tilde{x}}W = \ker D(QG)(\tilde{x})$ and the remark that (2.12) amounts to

$$\tilde{x} \in W^c \Rightarrow \ker \tilde{A}(\tilde{x})_{|T_{\tilde{x}}W} = \{0\}.$$

Now the result is a direct consequence of [RRh2, Theorem 4.1], except for the C^∞ - smoothness of the solution which is obvious from the proof given there when $\tilde{A}$ and $\tilde{G}$ are of class C^∞. Alternatively, the theorem also follows from [RRh1, Theorem 6.1] (in [RRh1], geometrically nonsingular DAE's are simply called "nonsingular"). $\square$

The terminology "geometrically nonsingular" in Definition 2.2 is justified by the fact that the various sets carrying the information needed in the existence theory are equipped with a "natural" differentiable structure (the proof of Theorem 2.1 eventually relies upon standard theory of ODE's on manifolds). Thus, the problem does not exhibit any visible (geometric) singularity. But Definition 2.2 also allows for invisible (or algebraic; see [RRh2]) singularities, as we now explain.

Suppose that the DAE (2.1) is geometrically nonsingular of index 1. From Theorem 2.1, no path $(t, x(t))$ goes through a point of the set $W \setminus W^c$ (closed in W), if x is a C^1 solution of (2.1). But such points may well lie at the "beginning" or the "end" of such a path $(t, x(t))$, for unlike in explicit ODE theory, here trajectories may stop abruptly at points reached in finite time. *Impasse points*, as defined below (Definition 2.3) are the most frequently encountered points of this type.

Impasse points are defined in [RRh2] for general quasilinear DAE's. Hence, in principle, a definition for an impasse point of (2.1) can be obtained by applying that definition to the

DAE (2.16) equivalent to (2.1). However, the general definition of impasse points makes reference to the rather unintuitive concept of the intrinsic derivative of some vector bundle morphism associated with the DAE. The definition we now give is an equivalent analytic translation of the abstract one in [RRh2], specialized to the DAE (2.16) - (2.17). The verification of this equivalence is a fairly technical exercise. However, the method used in the proof of [RRh2, Theorem 6.1], dealing with a special case, should give a reliable idea of the procedure to follow.

Let $(t,x) \in W$ be given, and suppose that

$$\dim\ker Q(t)D_xG(t,x)_{|\ker A(t)} = 1,$$

so that $(t,x) \in W \backslash W^c$ by condition (2.12). Evidently we have rank $Q(t)D_xG(t,x)_{|\ker A(t)} = r-1$, whence

$$\dim[\text{rge } Q(t)D_xG(t,x)_{|\ker A(t)}]^\perp \cap Z = 1.$$

Now, let $\tilde{u}$ be a nonzero vector in $[\text{rge } Q(t)D_xG(t,x)_{|\ker A(t)}]^\perp \cap Z$. Equivalently, $\tilde{u} \in Z$ and $[Q(t)D_xG(t,x)]^T\tilde{u} \in [\ker\ A(t)]^\perp = \text{ rge } A(t)^T$, so that there is a unique element $\tilde{\eta} \in \mathbb{R}^n$ such that

$$\tilde{\eta} \in \text{ rge } A(t), \quad A(t)^T\tilde{\eta} = [Q(t)D_xG(t,x)]^T\tilde{u}. \tag{2.19}$$

Remark 2.1: Observe that the condition $\tilde{u} \in [\text{rge } Q(t)D_xG(t,x)_{|\ker A(t)}]^\perp$, or equivalently, $\tilde{u} \in \ker(Q(t)D_xG(t,x)_{|\ker A(t)})^T$, in no way implies that $\tilde{u} \in [\text{rge } Q(t)D_xG(t,x)]^\perp$; i.e., $\tilde{u} \in \ker(Q(t)D_xG(t,x))^T$, as could inadvertantly be inferred. Thus, $\tilde{\eta}$ in (2.19) may, but need not, be 0. $\square$

These preliminaries lead to the following concept:

Definition 2.3. *The point $(t,x) \in W$ is an impasse point of the DAE (2.1) if*

$$\dim\ker Q(t)D_xG(t,x)_{|\ker A(t)} = 1, \tag{2.20}$$

and if

$$\langle Q(t)D_x^2G(t,x)(u)^2, \tilde{u}\rangle \neq 0 \tag{2.21}$$

holds for any pair of nonzero vectors

$$(u,\tilde{u}) \in \{\ker Q(t)D_xG(t,x)_{|_{\ker A(t)}}\} \times \{[\mathrm{rge}\ Q(t)D_xG(t,x)_{|_{\ker A(t)}}]^{\perp} \cap Z\}.$$

Remark 2.2: (i) Condition (2.21) is unaffected by the choice of the pair $(u, \tilde{u})$ since different choices amount to replacing u and $\tilde{u}$ by nonzero scalar multiples. (ii) Conditions (2.20) and (2.21) are also independent of the choice of the space Z in (2.3), as they are equivalent to the intrinsic conditions given in Definition 5.1 of [RRh2]. □

The definition of impasse points is incomplete without the notion of accessibility based on the following result:

Proposition 2.2. *Let (t, x) be an impasse point of (2.1) (hence $(t, x) \in W$) and let $\tilde{u}$ be as in Definition 2.3 and $\tilde{\eta}$ defined by (2.19). Then*

(i) the vector

$$(2.22) \qquad (\langle D_t(QG)(t,x), \tilde{u}\rangle, \tilde{\eta}) \in \mathbb{R} \times \ \mathrm{rge}\ A(t)$$

is nonzero and generates the orthogonal complement in $\mathbb{R} \times$ rge $A(t)$ of the space of vectors of the form $(\tau, A(t)h)$ with $(\tau, h) \in T_{(t,x)}W \subset \mathbb{R} \times \mathbb{R}^n$, and

(ii) we have

$$(2.23) \qquad \langle D_t(QG)(t,x), \tilde{u}\rangle + \langle G(t,x), \tilde{\eta}\rangle \neq 0.$$

Proof. (i) By contradiction, suppose that $\tilde{\eta} = 0$, i.e. $\tilde{u} \in \ker(Q(t)D_xG(t,x))^T$ by (2.19), and suppose that $\langle D_t(QG)(t,x), \tilde{u}\rangle = 0$. This implies that $\tilde{u} \in [\mathrm{rge}\ D(QG)(t,x)]^{\perp}$, and hence that $\tilde{u} = 0$ by condition (2.5) since $\tilde{u} \in Z$. This contradicts the assumption $\tilde{u} \neq 0$.

It is easily checked that the vector (2.22) is indeed orthogonal to all vectors $(\tau, A(t)h)$ with $(\tau, h) \in T_{(t,x)}W$, and it follows easily from condition (2.21) that this space has dimension $r - 1$, hence codimension 1 in $\mathbb{R} \times$ rge $A(t)$. This proves (i).

(ii) Once again by contradiction, suppose that

$$\langle D_t(QG)(t,x), \tilde{u}\rangle + \langle G(t,x), \tilde{\eta}\rangle = 0,$$

so that the vector $(1, G(t,x))$ is orthogonal to the vector (2.22). As $(t,x) \in W$, we have $Q(t)G(t,x) = 0$, i.e. $G(t,x) \in \operatorname{rge} A(t)$, and it follows from part (i) that $(1, G(t,x))$ has the form $(\tau, A(t)p)$ for some pair $(\tau, p) \in T_{(t,x)}W$. Obviously, $\tau = 1$ and hence $(t,x,1,p) \in TW$. On the other hand, since $G(t,x) = A(t)p$, we also have $(t,x,1,p) \in M$ (see (2.9)). Thus, $(t,x,1,p) \in TW \cap M$, whence $(t,x) \in \Pi(TW \cap M) = W^c$. But then $Q(t)D_xG(t,x)_{|\ker A(t)} \in GL(\ker A(t), Z)$ by condition (2.12), in contradiction with condition (2.20). □

From condition (2.21) and Proposition 2.2, the number

$$(2.24) \qquad (\langle D_t(QG)(t,x), \tilde{u}\rangle + \langle G(t,x), \tilde{\eta}\rangle)\langle Q(t)D_x^2G(t,x)(u)^2, \tilde{u}\rangle$$

is nonzero when (t,x) is an impasse point of the DAE (2.1). We shall say that (t,x) is *accessible* (resp. *inaccessible*) if the number (2.24) is *positive* (resp. *negative*). This makes sense because of the following result:

Proposition 2.3. *If $(t,x) \in W$ is an impasse point of (2.1), the sign of (2.24) is independent of the complement Z of rge $A(t)$, and independent of the choices of u and $\tilde{u}$ in $\ker Q(t)D_xG(t,x)_{|\ker A(t)} \setminus \{0\}$ and $[\operatorname{rge} Q(t)D_xG(t,x)_{|\ker A(t)}]^{\perp} \cap Z \setminus \{0\}$, respectively.*[3]

Proof. As various arguments are involved in this proof, we proceed in several steps.

(i) For fixed Z, the expression (2.24) is homogeneous of degree 2 in u, and also in $\tilde{u}$ since $\tilde{\eta}$ depends linearly upon $\tilde{u}$, so that its sign is independent of the choices of u and $\tilde{u}$. Moreover, the space $\ker Q(t)D_xG(t,x)_{|\ker A(t)} = \{h \in \ker A(t) : D_xG(t,x)h \in \operatorname{rge} A(t)\}$ is independent of Z, so that u may be fixed once and for all.

(ii) The expression (2.24) may be rewritten in a way making no use of the derivative $D_tQ(t)$: Since $(t,x) \in W$, there is $p \in \mathbb{R}^n$ such that $A(t)p = G(t,x)$, whence $D_tQ(t)G(t,x) = -Q(t)D_tA(t)p$ as was seen in the proof of Proposition 2.1. As a result, we have

$$\langle D_t(QG)(t,x), \tilde{u}\rangle = \langle Q(t)(D_tG(t,x) - D_tA(t)p), \tilde{u}\rangle.$$

[3]Recall that $\tilde{\eta}$ in (2.24) is uniquely determined by $\tilde{u}$ via (2.19).

(iii) Let $Z_0 := [\text{rge } A(t)]^\perp$ and call $Q_0(t)$ the corresponding (orthogonal in this case) projection operator. It is straightforward to check that $Q(t)^T \in \mathcal{L}(\mathbb{R}^n)$ is also a projection operator onto Z_0 (but not orthogonal unless $Z = Z_0$). As a result, $Q(t)^T\tilde{u} \in Z_0$. We claim that $Q(t)^T\tilde{u} \in \text{ rge } [Q_0(t)D_xG(t,x)_{|\ker A(t)}]^\perp$. Indeed, from the choice of $\tilde{u}$, we find $\langle \tilde{u}, Q(t)D_xG(t,x)h\rangle = 0$, for all $h \in \ker A(t)$, i.e. $\langle Q(t)^T\tilde{u}, D_xG(t,x)h\rangle = 0$, for all $h \in \ker A(t)$. As $Q_0(t)$ is the orthogonal projection onto $Z_0, Q_0(t) = Q_0(t)^T$ holds. Hence, using $Q(t)^T\tilde{u} = Q_0(t)Q(t)^T\tilde{u}$, we get $\langle Q(t)^T\tilde{u}, Q_0(t)D_xG(t,x)h\rangle = 0$, for all $h \in \ker A(t)$, which proves the claim. At this stage, we have that $\tilde{u}_0 := Q(t)^T\tilde{u}$ satisfies the condition $\tilde{u}_0 \in [\text{rge } Q_0(t)D_xG(t,x)_{|\ker A(t)}]^\perp \cap Z_0$. Also, from (2.19), $\tilde{\eta}$ coincides with the value $\tilde{\eta}_0$ obtained by choosing $Z = Z_0$ and $\tilde{u} = \tilde{u}_0$ in the first place. Using (ii) above twice (once with Z, once with Z_0) and the fact that $Q_0(t)$ is an orthogonal projector, it follows that (2.24) is unchanged when replacing $\tilde{u}$ by $\tilde{u}_0$. (In particular, $\tilde{u}_0 \neq 0$ since (2.24) is nonzero as noticed earlier.) This shows that in (2.24) we may replace Z by Z_0 upon replacing $\tilde{u}$ by $\tilde{u}_0$. But from (i), $\tilde{u}_0$ can be replaced by any scalar multiple without changing the sign of (2.24). Thus, the sign of (2.24) is independent of Z. This completes the proof. □

Because of (2.20) we have $(t,x) \in W \setminus W^c$ for any impasse point (t,x), and, hence, by Theorem 2.1(i), no C^1 solution of (2.1) may pass through (t,x). For this reason, we generalize the notion of a solution for the DAE (2.1) near impasse points.

Definition 2.4. *Let the DAE (2.1) be geometrically nonsingular of index* 1, *and let* (t_*, x_*) *be an impasse point of (2.1). A solution of (2.1) satisfying the condition* $x(t_*) = x_*$ *is a continuous function* $x : J_* \to \mathbb{R}^n$ *where* $J_* = [t_*, t_* + T)$ *or* $J_* = (t_* - T, t_*]$ *for some* $T > 0$, *such that* $x(t_*) = x_*$ *and* x *is a* C^1 *solution of (2.1) in* $J_*^0 = J_* \setminus \{t_*\}$.

Thus solutions of (2.1) in the sense of Definition 2.1 are "one-sided" and need not satisfy $A(t)\dot{x}(t) = G(t, x(t))$ for $t = t_*$. For the proof of the corresponding existence result, given below, we refer to [RRh2]).

Theorem 2.2. *Let* (t_*, x_*) *be an accessible (resp. inaccessible) impasse point of the geometrically nonsingular DAE (2.1) of index* 1. *There are exactly two solutions* $x(t)$ *of (2.1) in the sense of Definition 2.4 satisfying the conditions* $x(t_*) = x_*$, *and both are defined*

in $J_* = (t_* - T, t_*]$ (resp. $[t_*, t_* + T)$) for some $T > 0$. This result remains unaffected by shrinking $T > 0$ and both solutions are actually of class C^∞ in $J_*^0 = J_* \setminus \{t_*\}$. Moreover, $\lim_{t \to t_*} \|\dot{x}(t)\| = \infty$.[4]

This theorem justifies the terminology "impasse point", at least in the accessible case since the solutions cannot be continuously extended beyond t_*. As accessible points become inaccessible and vice-versa upon changing time evolution (i.e. changing t into $-t$), the terminology is justified in the inaccessible case as well. Note also that from Theorems 2.1 and 2.2, impasse points lie in the closure of W^c in W despite the fact that such a property is not explicitly incorporated into Definition 2.3. Finally, the result $\lim_{t \to t_*} \|\dot{x}(t)\| = \infty$ in Theorem 2.2 fully justifies dropping the requirement that $A(t)\dot{x}(t) = G(t, x(t))$ for $t = t_*$ in Definition 2.4.

Remark 2.3: In the proof of Theorem 3.2 in the next section, we shall make crucial use of a property more precise than $\lim_{t \to t_*} \|\dot{x}(t)\| = \infty$ in Theorem 2.2. A careful examination of the proof of [R, Theorem 5.1], from which Theorem 2.2 is derived in [RRh2], reveals that if (t_*, x_*) is (say) an accessible impasse point and $x : (t_* - T, t_*] \to \mathbb{R}^n$ denotes either of the two solutions of (2.1) in the sense of Definition 2.4 and satisfying $x(t_*) = x_*$, then $\|\dot{x}(t)\| = O((t_* - t)^{-1/2})$ for t near t_*. This shows that while $\dot{x}(t)$ blows up as t approaches t_*, nevertheless $\dot{x} \in (L^1(t_* - T, t_*))^n$. □

Remark 2.4 (Autonomous problems): When the time variable t does not enter explicitly in the DAE (2.1), it becomes involved rather artificially in the various concepts discussed earlier. For instance, with $A(t) = A$ and $G(t, x) = G(x)$, the set W in (2.6) becomes $W = \mathbb{R} \times \{x \in \mathbb{R}^n : G(x) \in \text{rge } A\}$. Here, the factor $\mathbb{R}$ is needed to take the variable t into account, but the only "important" factor is the set $\{x \in \mathbb{R}^n : G(x) \in \text{rge } A\}$. This suggests changing the notation for autonomous problems, thus eliminating the factor $\mathbb{R}$ and setting

$$W = \{x \in \mathbb{R}^n : G(x) \in \text{ rge } A\}. \tag{2.25}$$

[4] This statement differs from that in [RRh2, Theorem 5.1] where typographical errors resulted in an exchange of the roles played by accessible and inaccessible points.

Consistent with this new notation, the manifold M in (2.9) becomes

$$M = \{(x,p) \in \mathbb{R}^n \times \mathbb{R}^n : Ap - G(x) = 0\}.$$

In this case, the definition of the set of consistent points W^c given in (2.11) remains adequate (but, of course, looses its artificial time component). Likewise, time may be dropped from the definition of impasse points (Definition 2.3) since now the projection $Q(t)$ need not depend upon t. Hence, with W as in (2.25) above, we may and shall refer to $x(\in W)$ being an impasse point of the DAE $A\dot{x} = G(x)$ (so that x is an impasse point in this "new" sense if and only if (t,x) is an impasse point for every $t \in \mathbb{R}$ in the "old" sense). The accessibility/inaccessibility criterion remains based upon the sign of the quantity (2.24), now independent of t and hence reducing to $\langle G(x), \tilde{\eta}\rangle \langle QD_x^2 G(x)(u)^2, \tilde{u}\rangle$. Naturally, all the results discussed earlier have obvious analogs expressed in this new terminology. □

3. Discontinuous Solutions of Semilinear DAE's.

In the theory of explicit ODE's $\dot{x} = f(t,x)$ with smooth enough f, it is rather immaterial whether $\dot{x}$ should be viewed as the classical or the distribution derivative of x, except that the latter point of view introduces a few extra technicalities since $f(t,x)$ must be unambiguously defined for x in the chosen class of distributions. In particular, viewing $\dot{x}$ as the distribution derivative of x does not allow for the existence of new solutions in the class, say, of piecewise C^1 functions. Indeed, if x has "jumps", these get multiplied by Dirac delta distributions in $\dot{x}$, and hence must vanish for $\dot{x}$ to equal the *function* $f(t,x)$. It follows at once from this remark that piecewise C^1 solutions of $\dot{x} = f(t,x)$ are just its classical, C^1 solutions. We shall see here that things go quite differently for semilinear DAE's.

In the remainder of this section, we shall assume once and for all that the DAE (2.1) is geometrically nonsingular of index 1. The C^1 solutions of (2.1) about consistent points as well as the "one-sided" solutions of (2.1) about impasse points in the sense of Definition 2.4 will be referred to as "classical" solutions of (2.1). The solutions we shall be interested in here are only "piecewise classical" and may exhibit jumps at one or several points of

their domain of definition. Naturally, the consideration of such solutions dictates viewing $\dot{x}$ as a generalized derivative of x. Not surprisingly, we shall choose $\dot{x}$ to represent the derivative of x in the sense of distributions.

If $J = (a, b)$ and $x : J \to \mathbb{R}^n$ is a function of class C^1 in each subinterval $(a, t_0]$ and $[t_0, b)$ for some $t_0 \in J$, then $x \in (\mathcal{D}'(J))^n$ (distributions in J with values in $\mathbb{R}^n$) and, as is well-known

$$\dot{x} = (x_0^+ - x_0^-)\delta_{t_0} + \frac{dx}{dt}, \tag{3.1}$$

where x_0^- (resp. x_0^+) $= \lim_{t \to t_0^-} x(t)$ (resp. $\lim_{t \to t_0^+} x(t)$), δ_{t_0} is the Dirac delta distribution at t_0, and dx/dt denotes the function equal to the usual derivative of x in the union $(a, t_0) \cup (t_0, b) = J \setminus \{t_0\}$.

Remark 3.1: Formula (3.1) need *not* be true if x is C^1 in (a, t_0) and (t_0, b) and C^0 in $(a, t_0]$ and $[t_0, b)$, but it remains valid if x is ***absolutely continuous*** in $(a, t_0]$ and in $[t_0, b)$. This generalization will be crucial to the proof of Theorem 3.2 later. □

With $x : J \to \mathbb{R}^n$ being as before Remark 3.1, the function $G(t, x)$ also defines an element of $(\mathcal{D}'(J))^n$ in the obvious way. As a result, $A(t)\dot{x} - G(t, x)$ is a distribution on J with values in $\mathbb{R}^n$, and it makes sense to ask whether $A(t)\dot{x} - G(t, x) = 0$ in $(\mathcal{D}'(J))^n$, i.e. whether x solves the DAE (2.1) in the sense of distributions. A first answer is given next.

Theorem 3.1. *Let $J = (a, b)$ and let $(t_0, x_0^-), (t_0, x_0^+) \in W^c$ be given. After shrinking the interval J about t_0 if necessary, denote by x^- (resp. x^+) the unique C^1 solution of (2.1) in J satisfying the condition $x^-(t_0) = x_0^-$ (resp. $x^+(t_0) = x_0^+$), whose existence follows from Theorem 2.1, and set*

$$x(t) = \begin{cases} x^-(t) \quad , \; a < t < t_0, \\ x^+(t) \quad , \; t_0 < t < b. \end{cases} \tag{3.2}$$

Then, $x \in C^1((a, t_0]; \mathbb{R}^n) \cap C^1([t_0, b); \mathbb{R}^n)$ and x solves (2.1) in the sense of distributions if and only if

$$x_0^+ - x_0^- \in \ker A(t_0). \tag{3.3}$$

Proof. Trivial from formula (3.1) and the relation $A(t)\frac{dx}{dt} - G(t,x) = 0$ in $J \setminus \{t_0\}$. Evidently, condition (3.3) expresses the vanishing of the coefficient $A(t_0)(x_0^+ - x_0^-)$ of δ_{t_0} in the expression $A(t)\dot{x} - G(t,x)$. □

Interestingly, if (say) $(t_0, x_0^-) \in W^c$ is fixed in Theorem 3.1, condition (3.3) along with the requirement $(t_0, x_0^+) \in W^c$ imply that, in general, the possible values for x_0^+ form a *discrete* set. Indeed, since $\dim\ker A(t_0) = n - r$, condition (3.3) amounts to x_0^+ solving a system of r scalar equations, while the condition $(t_0, x_0^+) \in W$ (since $W^c \subset W$) holds if and only if $Q(t_0)G(t_0, x_0^+) = 0$, i.e. x_0^+ solves another set of $n - r$ scalar equations. Thus, in all, x_0^+ must solve a set of n scalar equations, usually independent of one another and hence having only isolated solutions. Note that by openness of W^c in W (see Proposition 2.1) the more stringent condition $(t_0, x_0^+) \in W^c$ may rule out some of the solutions x_0^+ but does not place any further limitation upon the dimensionality of the set of solutions, so that "generically" this set should remain discrete. Furthermore, while the choice $x_0^+ = x_0^-$ is always available, the nonlinear nature of the problem makes it possible for solutions with $x_0^+ \neq x_0^-$ to exist.

Remark 3.2: If the DAE (2.1) is *linear*, i.e. of the form

$$A(t)\dot{x} + B(t)x = b(t), \tag{3.4}$$

with A, B and b of class C^∞ so as to fit into the setting of this paper, then jumps do not occur and the only solutions of (3.4) in the sense of distributions are the classical ones (see [RRh4]). Assuming the coefficients A and B are smooth, discontinuous solutions of (3.4) can be obtained only if the right-hand side $b(t)$ itself is discontinuous, at least in the index 1 case considered here. Some generalizations will be discussed in Section 4. In contrast, nonlinearity alone is responsible for the existence of the discontinuous solutions obtained in Theorem 3.1. This is well illustrated by Example 3.1 below. □

Example 3.1: For $n = 2$ consider the autonomous DAE of the form (2.1) with

$$A(t) = A = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \quad G(t,x) = G(x) = \begin{pmatrix} 1 \\ x_1 + x_2 - x_2^3 \end{pmatrix}, \quad x \in \mathbb{R}^2. \tag{3.5}$$

It is straightforward to check that (3.5) is geometrically nonsingular of index 1 with, in the simplified notation for autonomous problems discussed in Remark 2.4,

$$W = \{(x_2^3 - x_2, x_2),\ x_2 \in \mathbb{R}\},\ W^c = W \setminus \left\{ \left(\tfrac{2\sqrt{3}}{9}, -\tfrac{\sqrt{3}}{3}\right),\ \left(-\tfrac{2\sqrt{3}}{9}, \tfrac{\sqrt{3}}{3}\right) \right\}.$$

The points $\xi_1 = (2\sqrt{3}/9, -\sqrt{3}/3)$ and $\xi_2 = (-2\sqrt{3}/9, \sqrt{3}/3)$ are accessible and inaccessible impasse points for (3.5), respectively.

Figure 3.1 gives a plot of W, the impasse points ξ_1, ξ_2, and of some point $x^{in} = (x_1^{in}, x_2^{in}) \in W^c$ that we will choose as initial value $x^-(0)$ for the C^1 solution x^- of (3.5). This solution is defined until it reaches the accessible point ξ_1; that is, its interval of definition is $(-\infty, 2\sqrt{3}/9 - x_1^{in})$. The arrows in Figure 3.1 represent the direction of evolution of the solutions of (3.5) as time increases (consistent with the accessible/inaccessible nature of ξ_1 and ξ_2, respectively).

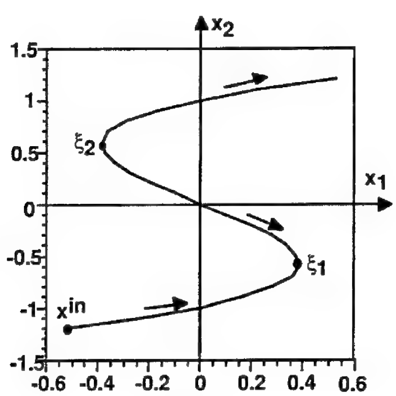

FIGURE 3.1

According to Theorem 3.1, jumps can only occur in the direction of the null-space of A, i.e., in the "vertical" direction. As a result, the solution x^- cannot be pieced together with another C^1 solution x^+ before $x_1^-(t)$ reaches the interval $(-2\sqrt{3}/9, 2\sqrt{3}/9)$; that is, before $t \in \mathcal{I} := (-2\sqrt{3}/9 - x_1^{in} 2\sqrt{3}/9 - x_2^{in})$. Indeed for $t_0 < -2\sqrt{3}/9 - x_1^{in}$ the vertical line $x_1 = x_1^-(t_0)$ intersects W only at the point $x^-(t_0)$; but as soon as $t_0 \in \mathcal{I}$, this line

intersects W at $x^-(t_0) := x_0^-$ and two other points. Either one can be chosen as initial condition x_0^+ at time t_0 for a C^1 solution of (3.5), and then formula (3.2) defines a solution of (3.5) in the sense of distributions exhibiting the jump $x_0^+ - x_0^-$ at time t_0. Since $t_0 \in \mathcal{I}$ was arbitrary, infinitely many (even uncountably many) distinct discontinuous solutions may be obtained by this process. Furthermore, after a first jump at t_0, the solution may jump again at any time $t_1 \in \mathcal{I}$ with $t_1 > t_0$ and evidently may even do so infinitely many times, in infinitely many ways. This is illustrated in Figure 3.2.

It is important to observe that all these discontinuous solutions emanate from the same point x^{in} at $t = 0$. In other words, *uniqueness of solutions* of initial value problems associated with (3.2) *breaks down completely* in the class of discontinuous solutions. This rather disturbing remark – obviously not limited to the specific example (3.5) – motivated the material contained in Part II of this article.

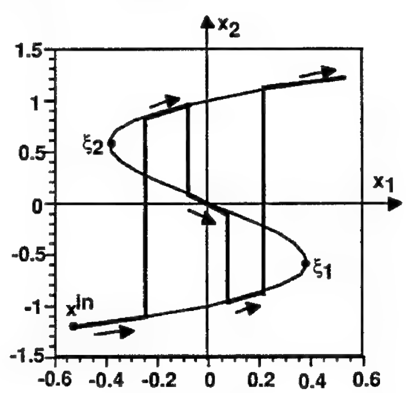

FIGURE 3.2

A pleasant feature about discontinuous solutions is that they provide the only reasonable way of getting out of the dead end created by accessible impasse points. Indeed, since classical solutions cannot be continued beyond accessible impasse points (Theorem 2.2) they *must* jump to proceed further. That this may be possible for solutions of (2.1) in the sense of distributions is shown in the subsequent generalization of Theorem 3.1.

Theorem 3.2. *Theorem 3.1 remains valid with the following modifications of its hypotheses: (t_0, x_0^-) is an accessible impasse point of the DAE (2.1), x^- is either of the two solutions of (2.1) satisfying $x^-(t_0) = x_0^-$ (Theorem 2.2) and the interval $J = (a, b)$ is such that x^- is defined in $(a, t_0]$.*

Proof. The arguments of the proof of Theorem 3.1 can be repeated verbatim, but the use of formula (3.1) must be justified, which is done via Remarks 3.1 and 2.3. □

The proof of Theorem 3.2 given above is deceptively short, as all the technicalities are contained in the results quoted in Remark 2.3. There is, of course, also an analog of Theorem 3.2 when (t_0, x_0^+) is an inaccessible impasse point, and one when (t_0, x_0^-) is an accessible impasse point and (t_0, x_0^+) an inaccessible impasse point.

Remark 3.3: By Theorem 3.2, jumps at the impasse point (t_0, x_0^-) must occur in the null-space $\ker A(t_0)$. This result agrees with Takens' assumption in his work [T] on discontinuous solutions for special cases of (2.1) (autonomous with a gradient structure) and also with the work of Sastry and Desoer [SDe]. But neither Takens nor Sastry and Desoer ever consider distribution solutions, and they use other arguments to justify their concepts of solution. Also, in these approaches, jumps are allowed to occur only at singularities. (In [SDe], it is observed that jumps could also occur at other points, but this possibility is next ruled out by the introduction of extra assumptions.) □

It should be pointed out that Theorem 3.2 does not state that jumps *always* allow the solutions to be continued beyond impasse points as is obviously the case for Example 3.1. For example, Figure 3.3 below relates to the DAE

$$\begin{cases} \dot{x}_1 = 1, \\ 0 = x_1 + x_2^2, \end{cases}$$

for which A is as in (3.5) and $W = \{(-x_2^2, x_2); x_2 \in \mathbb{R}\}$ (using once again the "autonomous" notation of Remark 2.4). In this case, jumps from impasse points cannot occur. In fact, $x_0^- = (0, 0)$, is the only (accessible) impasse point and there is no point $x_0^+ \in W^c$ such that $x_0^+ - x_0^- \in \ker A$, and hence no point to which a solution reaching x_0^- can jump.

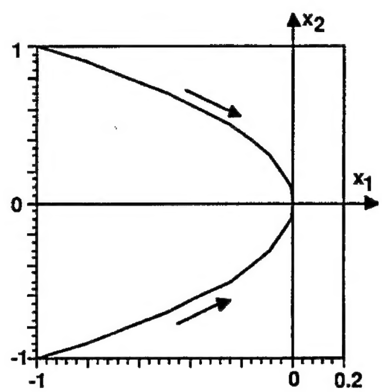

FIGURE 3.3

4. Forced Discontinuities and Inconsistent Initial Conditions.

The discontinuous solutions discussed in the previous section are in some sense "self-generated", as they exist from only the combined effect of the nonlinearity of $G(t, x)$ with respect to the x variable and the weaker requirement that (2.1) be understood in the sense of distributions. As noted in Remark 3.2, such self-generated discontinuities do not exist in linear problems.

A different situation, also frequently encountered in practical applications, is that the DAE governing the system suddenly changes at a given time t_0. Actually, this situation always presents itself when the DAE (2.1) begins to govern the system at time t_0, and the state variable has evolved in an unrelated way for $t < t_0$. Mathematically, we may assume that this "past" history is accounted for by a *known* function $x^-(t)$ for $t < t_0$ and that $\lim_{t \to t_0^-} x^-(t) := x_0^-$. Evidently, x_0^- is a natural initial condition to associate with the DAE (2.1) if the latter is to describe the state of the system for $t \geq t_0$. The only problem of course is that there is no reason why (t_0, x_0^-) should satisfy the consistency condition $(t_0, x_0^-) \in W^c$ which is necessary for (2.1) to have a classical solution satisfying $x(t_0) = x_0^-$. This directly leads to the well known problem of *inconsistent initial* conditions. This

problem is unsolvable in the framework of classical solutions, and has a straightforward answer when discontinuities are permitted and solutions are understood in the sense of distributions.

Indeed, the problem formulated above is one of extending a known function $x^-(t)$ into a solution of (2.1). Let $b(t)$ be the function

$$b(t) = \begin{cases} A(t)\dfrac{dx^-}{dt}(t) - G(t, x^-(t)) & , \text{ for } a < t < t_0 \\ 0 \text{ for } t_0 < t < b, \end{cases} \tag{4.1}$$

where dx^-/dt denotes the classical derivative of x^- (assuming x^- of class C^1 in $(a, t_0]$) and consider the problem of finding a function x of class C^1 in $(a, t_0]$ and in $[t_0, b)$, satisfying

$$\begin{cases} A(t)\dot{x} - G(t, x) = b(t) \text{ in } (\mathcal{D}'(J))^n, \\ x_{|(a,t_0)} = x^-. \end{cases} \tag{4.2}$$

It is easily checked that x solves (4.2) if and only if $x(t) = x^-(t)$ for $t < t_0$, and $x_{|[t_0,b)} := x^+$ solves

$$\begin{cases} A(t)\dfrac{dx^+}{dt} - G(t, x^+) = 0 \\ x^+(t_0) = x^+, \end{cases} \tag{4.3}$$

with $x_0^+ - x_0^- \in \ker A(t_0)$ (so that no Dirac delta distribution appears in (4.2)). As in the previous section, we thus have that x_0^+ is determined from x_0^- by the two conditions $x_0^+ - x_0^- \in \ker A(t_0)$ and $Q(t_0)G(t_0, x_0^+) = 0$, the latter for consistency of (t_0, x_0^+) with the DAE (2.1) (i.e. (4.3)). By nonlinearity of G, x_0^+ is usually non-unique. On the other hand, from the fact that the conditions $x_0^+ - x_0^- \in \ker A(t_0)$ and $Q(t_0)G(t_0, x_0^+) = 0$ represent a system of n equations in n scalar unknowns for x_0^+, only isolated solutions should be expected. These considerations provide a simple satisfactory answer to the problem of inconsistent initial conditions (although not quite complete since x_0^+ need not be uniquely determined). Further limitations on the choice of x_0^+ will be dictated by the results in Part II of this article.

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There is strong physical evidence that a full treatment of differential-algebraic equations should be incorporate solutions with jump discontinuities. It is shown here that for semilinear problems the setting of distributions allows for the development of a theory where indeed such discontinuities may occur. This approach also settles the problem of inconsistent initial conditions in a very simple way. On the other hand, new issues arise as not only uniqueness, but even countability of the number of solutions of initial value problems may now be lost. A physically motivated but purely mathematical selection procedure to overcome this difficulty is discussed in Part II of this paper.

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